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The Fast Fourier Transform and the use of the Computer Type 7504 and the Digital Event Recorder Type 7502 as a Fast Fourier Transform Analyzer

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THE FAST FOURIER TRANSFORM AND THE USE OF THE COMPUTER TYPE 7504 AND THE DIGITAL EVENT RECORDER TYPE 7502 AS A FAST FOURIER TRANSFORM ANALYZER

By Arne Grøndahl and Roger Upton

INTRODUCTION

The idea of the Fast Fourier Transform (FFT) first appeared in 1965. It provides an algorithm which gives a far faster computation of the Discrete Fourier Transform than was previously possible, and has created for it a complete new range of applications where its use was previously thought impracticable. This Application Note starts off with a brief look at the philosophy behind FFT, and then goes on to describe a practical application of it using a Computer Type 7504 and a Digital Event Recorder Type 7502 as a single channel FFT Analyzer.

1. THE FOURIER TRANSFORM, THE DESCRETE FOURIER TRANSFORM, AND THE FAST FOURIER TRANSFORM

The Fourier Transform pair is defined as follows:

$$X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j\, 2\pi ft)\, dt \qquad (1)$$

$$x(t) = \int_{-\infty}^{\infty} X(f) \exp(j\, 2\pi ft)\, df \qquad (2)$$

Since its conception in the early years of the nineteenth century, it has become an almost indispensible tool throughout the fields of physics and engineering, the two equations allowing the transfer from the time domain to the frequency domain and back again to be carried out as desired. However, when a waveform is being sampled, or when a system is being analyzed on a digital computer, the classic equations, given above, can no longer be used. They are of a continuous nature, and are therefore quite unsuitable for digital processing. Hence, the question arises as to how these equations can be used with modern digital processing techniques.

The basic answer to this problem is obvious: equations (1) and (2) must be converted from a continuous form to a discrete form. Looking now at equation (1) only, one method of doing this is to replace the integral with a summation, and to limit the frequency variation to certain discrete values. Hence, if the waveform is sampled N times in a time T:

$$X(fn) = \Delta T \sum_{k=0}^{N-1} x(t_k) \exp(-j\, 2\pi f_n t_k) \qquad (3)$$

where $n = 0, \pm 1, \pm 2, \ldots \pm N/2$

Equation (3) can be simplified a little by noting that $\Delta T = T/N$ and $\Delta f = 1/T$, and by letting $t_k = k\Delta T$ and $f_n = n\Delta f$. Hence:

$$X(n) = \Delta T \sum_{k=0}^{N-1} x(k) \exp(-j\, 2\pi \tfrac{nk}{N}) \qquad (4)$$

where $n = 0, 1, 2, \ldots N-1$.

Similarly for equation (2)

$$x(k) = \Delta f \sum_{n=0}^{N-1} X(n) \exp(j\, 2\pi \tfrac{nk}{N}) \qquad (5)$$

where k $= 0, 1, 2, \ldots\ldots\ldots\ldots N-1$

Equations (4) and (5) make up the Discrete Fourier Transform (DFT) pair. Given N samples of the wave form, equation (4) gives N samples of the spectrum, and given N samples of the spectrum, equation (5) gives N samples of the waveform. Hence the problem of digital Fourier transform processing would appear to have been solved.

If we let $W = \Delta T \exp \left[-(j\,2\pi)/N\right]$ then

equation (4) becomes:

$$X(n) = \sum_{k=0}^{N-1} x(k)\, W^{nk} \qquad (6)$$

Equation (6) can conveniently be converted to matrix form such that:

$$\left[X(n)\right] = \left[W^{nk}\right]\left[X_o(k)\right] \qquad (7)$$

Suppose that now we let N be equal to 4, that is we have taken 4 samples of a waveform and we require 4 samples of its spectrum. Writing out equation (7) in full for this case yields:

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W^o & W^o & W^o & W^o \\ W^o & W^1 & W^2 & W^3 \\ W^o & W^2 & W^4 & W^6 \\ W^o & W^3 & W^6 & W^9 \end{bmatrix} \begin{bmatrix} x_o(0) \\ x_o(1) \\ x_o(2) \\ x_o(3) \end{bmatrix} \qquad (8)$$

It is here that the fundamental problem with the direct evaluation of the DFT appears. To evaluate equation (8) will take 16 complex (W is complex) multiply and add operations. Put into more general terms, the direct evaluation of an N sample (or N - point) DFT will take N^2 complex multiply and add operations. This makes it a highly expensive and inefficient piece of digital processing. Hence, although direct evaluation of the DFT represents one solution to the problem of digital Fourier transform processing, it is a solution which is hardly worth using.

The above situation meant that the use of Fourier transforms in digital processing was avoided. In 1965, however, the Cooley – Tukey FFT algorithm was published, (reference 3), and this revolutionised the whole approach to digital Fourier transform processing. It is an alternative method of evaluating the DFT which gives a startling reduction in the number of required operations. It can be considered as being a means of factorising the W^{nk} matrix of equation (7) such that the number of operations involved in its multiplication with the X_o (k) matrix is reduced.

When using the FFT, it is convenient to choose the number of sample points (i.e., N) such that they are an exact power of two. In equation (8), $N = 2^2$, and hence the FFT can be applied.

The first step is to write each W^{nk} term in (8) in the manner $W^{nk \bmod N}$. Hence since N = 4:

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W^1 & W^2 & W^3 \\ 1 & W^2 & W^0 & W^2 \\ 1 & W^3 & W^2 & W^1 \end{bmatrix} \begin{bmatrix} x_0(0) \\ x_0(1) \\ x_0(2) \\ x_0(3) \end{bmatrix} \qquad (9)$$

Equation (9) can then be factorised as follows:

$$\begin{bmatrix} X(0) \\ X(2) \\ X(1) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & W^0 & 0 & 0 \\ 1 & W^2 & 0 & 0 \\ 0 & 0 & 1 & W^1 \\ 0 & 0 & 1 & W^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & W^0 & 0 \\ 0 & 1 & 0 & W^0 \\ 1 & 0 & W^2 & 0 \\ 0 & 1 & 0 & W^2 \end{bmatrix} \begin{bmatrix} x_0(0) \\ x_0(1) \\ x_0(2) \\ x_0(3) \end{bmatrix} \qquad (10)$$

If we recognise that $W^0 = - W^2$, the evaluation of equation (10) will take 4 complex multiplications and 8 complex additions. This compares with 16 complex multiplications and additions in the case of equation (8). Hence, a considerable saving in the number of operations required has been accomplished.

Put into more general terms, the Cooley-Tukey algorithm effectively factorises an N x N matrix into a(N x N) matrices, where $N = 2^a$. The greater N is, then the greater the saving in operations in relation to the number required for direct evaluation of the DFT. The saving in computer time obtained by using the algorithm is approximately in the ratio a/N. Hence, for a 2^{10} point transform, FFT is approximately 100 times faster than the direct evaluation of the DFT.

Further information may be found, if required, from references (1) and (2), which are both of an introductory nature. These, in turn, also contain further valuable references.

2. THE DIGITAL EVENT RECORDER TYPE 7502 AND THE COMPUTER TYPE 7504 AS AN FFT ANALYZER

Although not specifically designed as such, the Digital Event Recorder Type 7502 and the Computer Type 7504 can be operated together as a single channel FFT Analyzer. Such an Analyzer is illustrated in Fig.1. In it, the 7502 acts as the capture and display part and A/D and D/A Converter while the 7504 carries out all the processing. The Teletypewriter Type 6401 forms the basic means of communication between the Analyzer and its operator. It is also used for program entry and as an alternative means of data input and output. Where its rate of input and output is too Slow, however a Tape Punch Type 6301 and a Tape Reader Type 7102 may be added to the System.

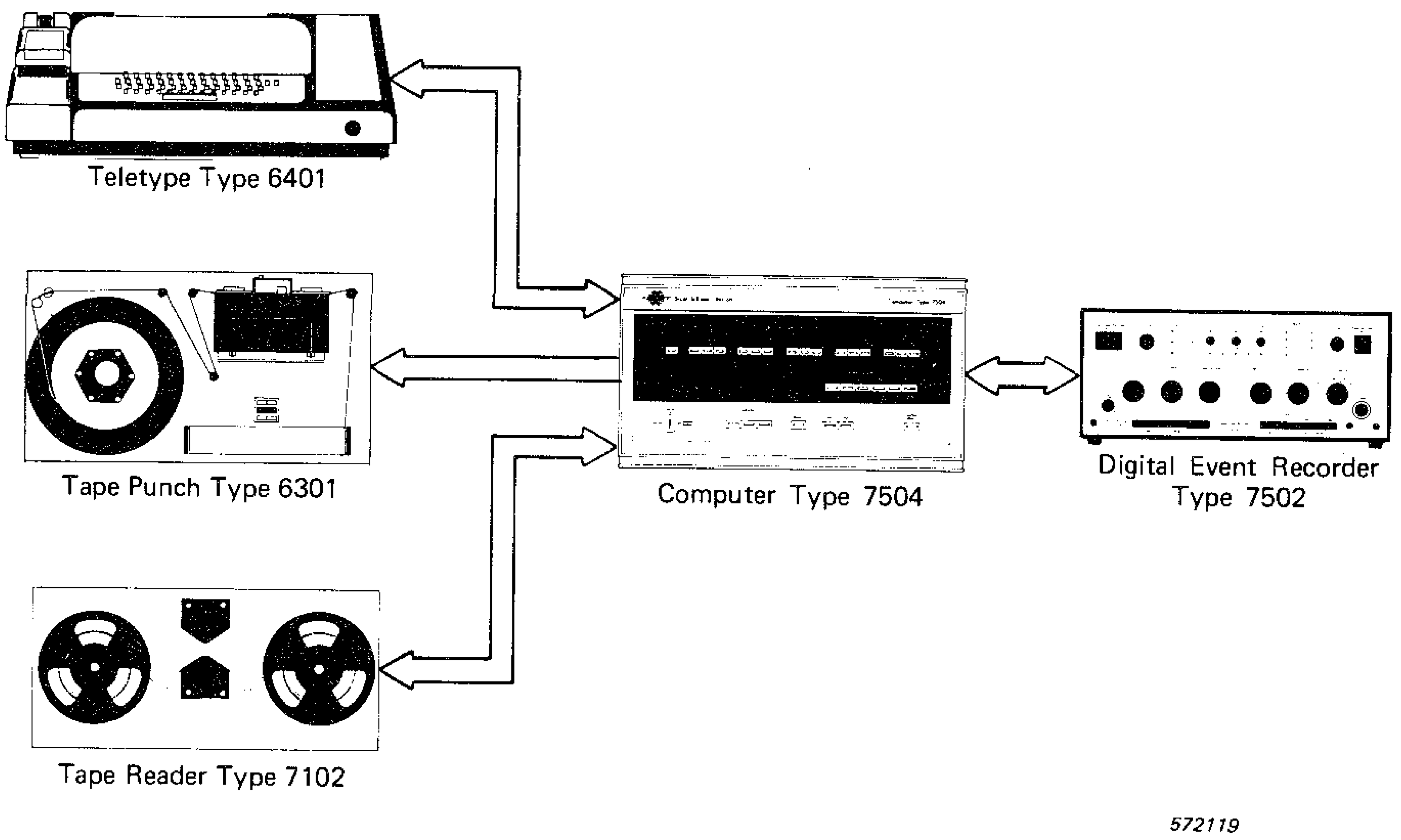


Fig. 1. FFT Analyzer comprising of a Computer Type 7504 and a Digital Event Recorder Type 7502

The normal size of Memory of the 7502 when used in this application is 4k, (although any other size could equally well be used). Likewise, the normal size of Memory for the 7504 is 8k, this being split into 4k for data and results storage, and 4k for program storage and processing. Hence, where the processing function being carried out requires FFT on only one function (e.g., computing the voltage or power spectrum of a time function), the number of points in the transform can be any power of 2 up to 2^{12} (= 4096 = 4k). Where, however, it requires FFT to be carried out on two functions (e.g., computing or transfer function or an crosscorrelation function), the amount of Memory in the 7504 available for data storage (4k) limits the maximum number of points in each transform to 2^{11} (= 2048 = 2k) unless external data storage or a 12k 7504 is used.

The complete software package for the analyzer allows the following operations to be carried out:

Forward and inverse Fourier transform (1024 points in 0.3 s);
Power Spectrum measurement;
Autocorrelation Function measurement;
Hanning, Hamming and other weighting functions;
Ensemble averaging in time or frequency domain;
Complex and Complex conjugate multiplication;
Standard arithmetic operations (add, subtract, multiply, divide);
Coordinate conversion (polar/rectangular, i.e., real and imaginary part to amplitude and phase);
Logarithmic/Linear conversion:

Each of these subroutines may be carried out separately, being commanded from the 6401, or alternatively, they may be combined into a single complex program.

An n point FFT is performed on a time function by first capturing it on the 7502. Any aliasing problems which can be caused by the sampling of the function by the 7502 are taken care of by the 7502's inbuilt antialiasing filters, which cut off at a quarter of the sampling frequency. Hence, assuming a 4k 7502, 4k points of the time function, having a upper limiting frequency of a quarter of the sampling frequency, will be contained in the 7502's Memory. To perform the n point transform, n of the 4k points contained in the 7502's Memory are transferred to the 7504. When n is less than 4k, they may be taken from anywhere within the 4k, as long as they are consecutive. The 7504 then performs the FFT Algorithm on the n points, producing a result which has an (n/2) + 1 point real part and an (n/2) −1 imaginary part. This result is stored in the Memory of the 7504, such that it may later be manipulated with another result to give, e.g., a transfer function, or such that it can be read out and displayed in terms of e.g., amplitude and phase. The actual display of a result is achieved by its read-out to the 7502 Memory. A Play Back to a device such as an oscilloscope or a Level Recorder is then all that is required to complete the process.

An n point transform (or n point transforms where two are being used to produce a transfer function or autocorrelation function) will always produce an n point result. However, the result is in two separate parts, each part consisting of ≅ n/2 points. Further, in the forward transform (i.e., from time to frequency), each of these parts has an upper limiting frequency, this being half the frequency with which the time function was originally sampled (i.e., the Nyquist or folding frequency). Hence, the resolution of each part in the frequency domain is this frequency divided by n/2. This means that the amount of detail seen in a transform can be increased by increasing the number of points, thus increasing the resolution. There is a limit to this, though, beyond which no matter how many points are used, no further information is seen. This is the reciprocal of the length in the time domain represented by the event being transformed and is known as the bandwidth. For instance, if an event is 100 ms long, on transformation, it will have a bandwidth of 10 Hz. This means that no matter how many points are taken, no more information can be obtained from the transform than that obtained with a 10 Hz resolution. E.g. a 5 Hz resolution would give no more information than a 10 Hz resolution, although the information would be presented in more detail.

Note that where the antialiasing filters are used on the 7502, although the results will be from DC to half the sampling frequency, those above a quarter of the sampling frequency will not be usable. Hence, the software ensures that they are not read out by the 7504 into the Memory of the 7502, and the frequency range of the displayed result is DC to a quarter of the sampling frequency. The resolution and bandwidth of the read-out is, however, unchanged. Note also that with analyses which normally use two simultaneously recorded channels of information, e.g., transfer function and crosscorrelation measurement, it is only possible to do them on this Analyzer if it is possible to repeat stimulii etc. in a manner such that they are synchronised with the sampling instants. It would, however, of course be possible to convert the system into a two channel analyzer by using two 7502s in Master-Slave configuration.

3. USE OF FFT FOR THE MEASUREMENT OF LOUDSPEAKER TRANSFER FUNCTIONS

The conventional method of measuring the transfer function of a loudspeaker is to place it in an anechoic chamber, and to excite it using a sine wave. This sine wave is then swept at constant amplitude through the frequency range of interest. The response of the loudspeaker to the sine wave is picked up with a microphone, and its variation with frequency is traced out using an instrument such as a Level Recorder. This gives the magnitude of the transfer function. Its phase is measured using a phase meter, its variation with frequency again being traced out with an instrument such as a Level Recorder.

This method has two disadvantages, first, that it requires an anechoic chamber, and second, that it takes a long time. The reason behind them is that the loudspeaker is only ever excited with one frequency at a time. An alternative to this would be to excite the loudspeaker with many frequencies at a time, and this is the idea behind the method using FFT.

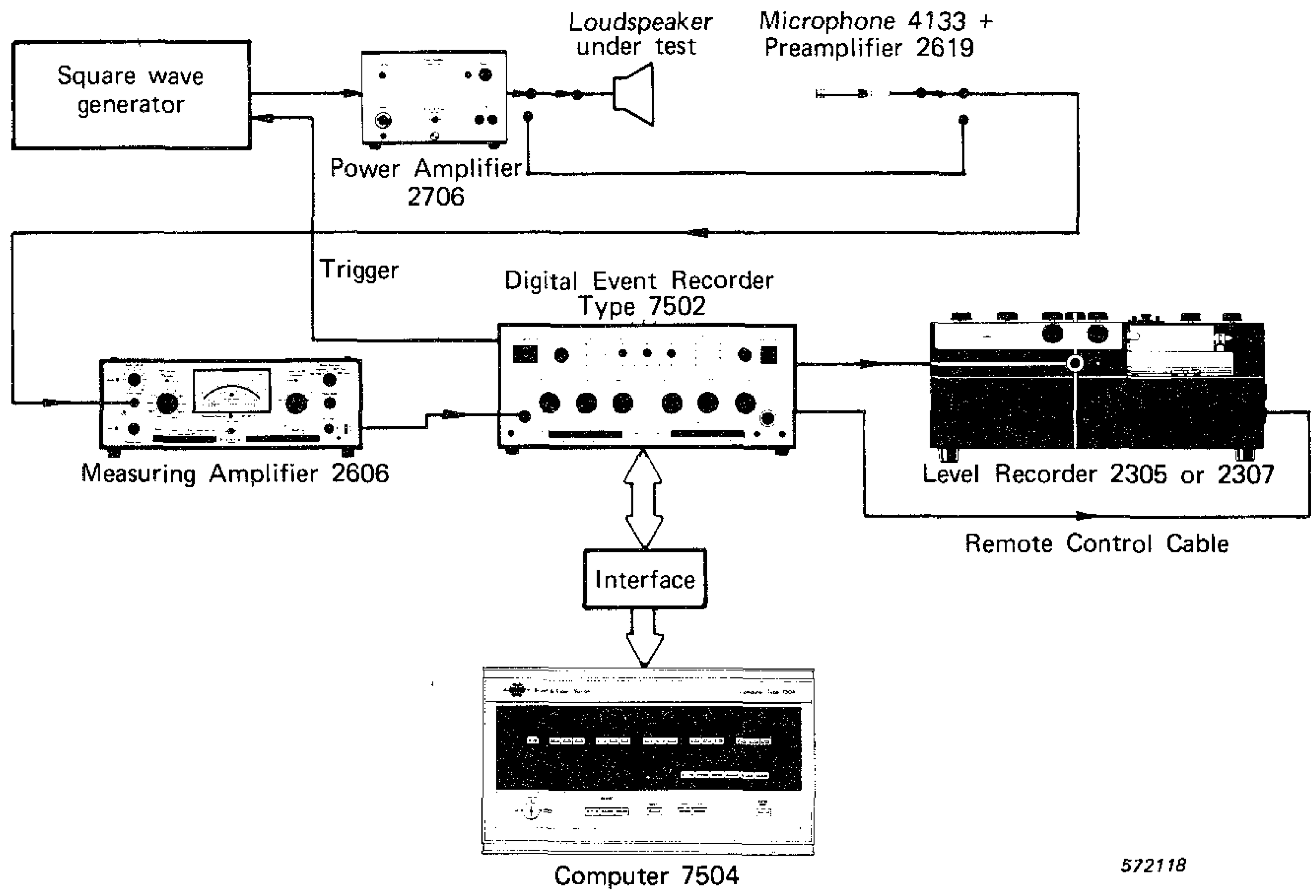


Fig.2. Experimental set-up for measuring Loudspeaker Transfer Functions

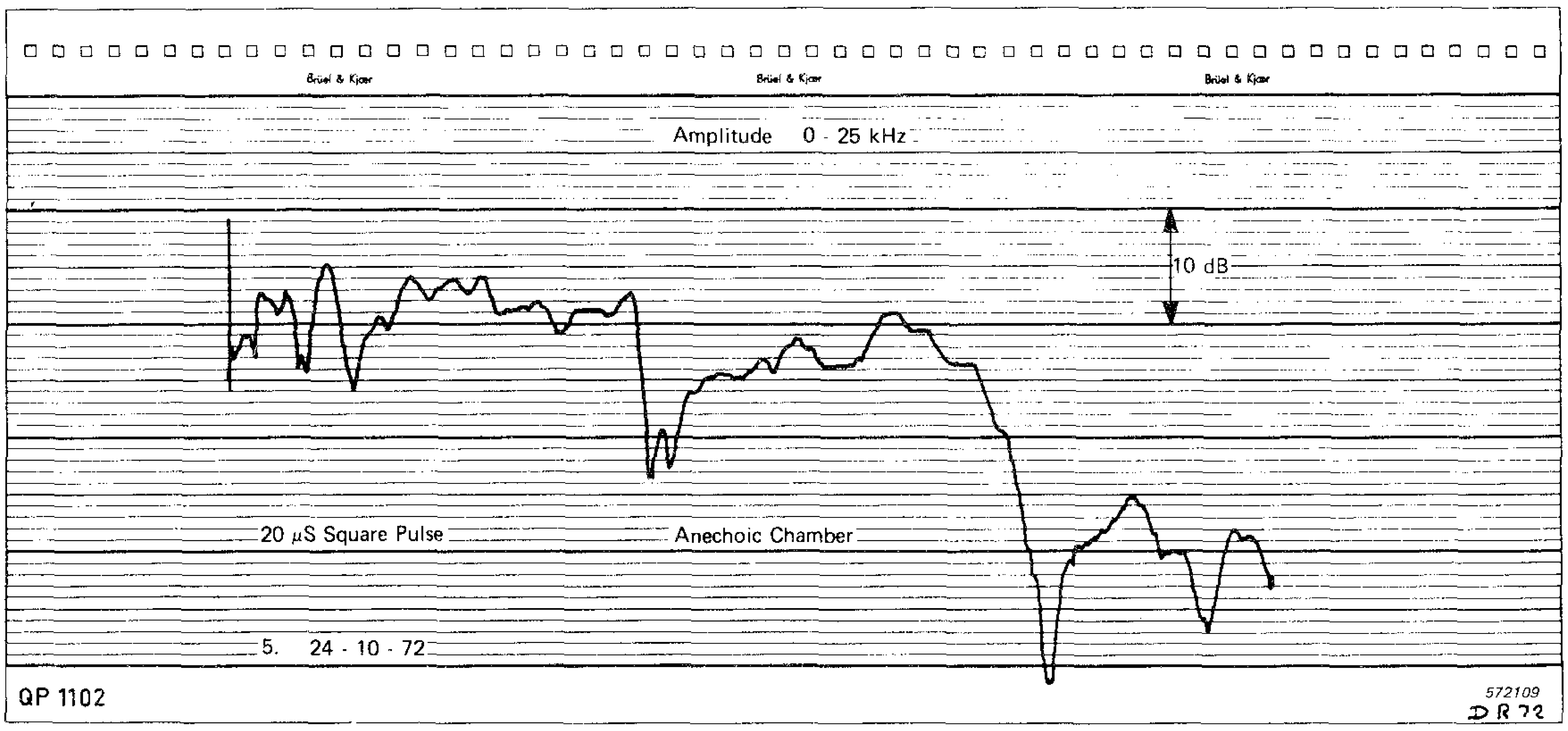
The system used to measure the transfer function of a loudspeaker using FFT is shown in Fig.2. In it, the 7502 is fully interfaced into the 7504, its functions being under the control of the 7504 software. It is also slightly modified, in that it triggers itself immediately as it is put into "Record" mode.

The principle of the test is that instead of being excited with one frequency at a time, in the form of a sinusoid, the loudspeaker is excited with an infinite number of frequencies, in the form of a pulse. The ideal pulse to use would be one which had a flat frequency spectrum up to the highest frequency of interest in the test, and zero content thereafter. However, such an ideal pulse cannot be realised. For a real pulse, the requirements are that its spectrum is non-zero throughout the frequency range of interest, and that it also has a reasonable signal level in this range. Probably the best pulse to use is a Sin^2 x pulse, but in the tests following, a square pulse was used. The spectrum of the square pulse is the well known (Sin x)/x function, with zeros appearing at multiples of the reciprocal of the pulse width. Hence, if the pulse width is set to the order of half the reciprocal of the highest frequency of interest, the two requirements will be satisfied. In the test, two pulse widths are used, namely, 20 μs to give the transfer function from 0 – 25 kHz and 200 μs to give it from 0 – 2.5 kHz.

The sampling rates used on the 7502 were 100 kS/s and 10 kS/s, giving FFTs with upper limiting frequencies of 50 kHz and 5 kHz respectively. However, as explained previously, the fact that antialiasing filters are used to avoid aliasing problems, they are really only usable up to 25 kHz and 2.5 kHz. Hence, 20 μs and 200 μs pulses, which have their first zeros at 50 kHz and 5 kHz respectively, are reasonable.

The process involved in the test is fairly simple. A command from the 7504 puts the 7502 into "Record" mode, which, because of the modification, causes it to be triggered immediately.

At the same time, the Pulse Generator is triggered, and emits the required square pulse. The pulse itself, or the loudspeaker response to it, (depending on whether or not the loudspeaker and microphone are by passed), is then Recorded on the 7502, whereby it is transmitted to the 7504 and the FFT algorithm performed on it. The result is stored on the 7504. The test is performed twice, once with the loudspeaker and microphone bypassed, and once with them in circuit. From there on, using the stored results, it is a simple matter to compute the required loudspeaker transfer function.



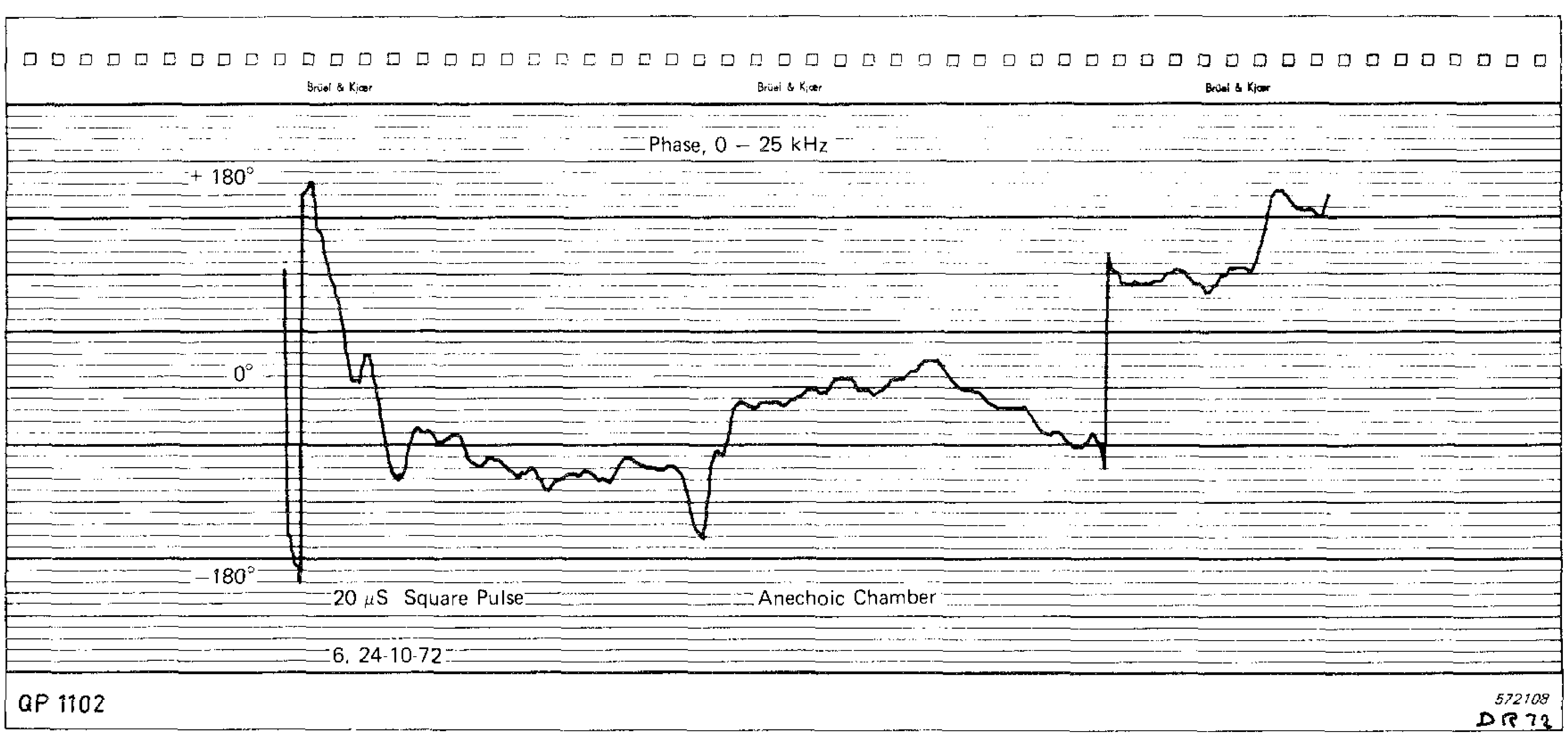


Fig.3. Amplitude and Phase Characteristics measured using 20 μs Square Pulse in Anechoic Chamber

Fig.3 shows the amplitude and phase of the transfer function of a loudspeaker obtained using this method. The frequency scale is 0 – 25 kHz (as explained previously), the 20 μs pulse being used. 2k point transforms were used with a 100 kS/s sampling rate, and hence the resolution is 50 Hz. (= 50 kHz/1k). The length of time represented by the points is ~ 20 ms (= 2k x 100 kS/s), and hence the bandwidth is also 50 Hz. The results were obtained using ensemble averaging in the time domain before the FFT Algorithm was performed, this process being carried out entirely under the software control of the 7504. After the results had been obtained, the test was repeated using a pulse width of 200 μs, the 7502 being set to Record at 10 kS/s. The amplitude and phase of the transfer function from 0 – 2.5 kHz was hence obtained, (Fig.4) resolved to 5 Hz in the frequency domain.

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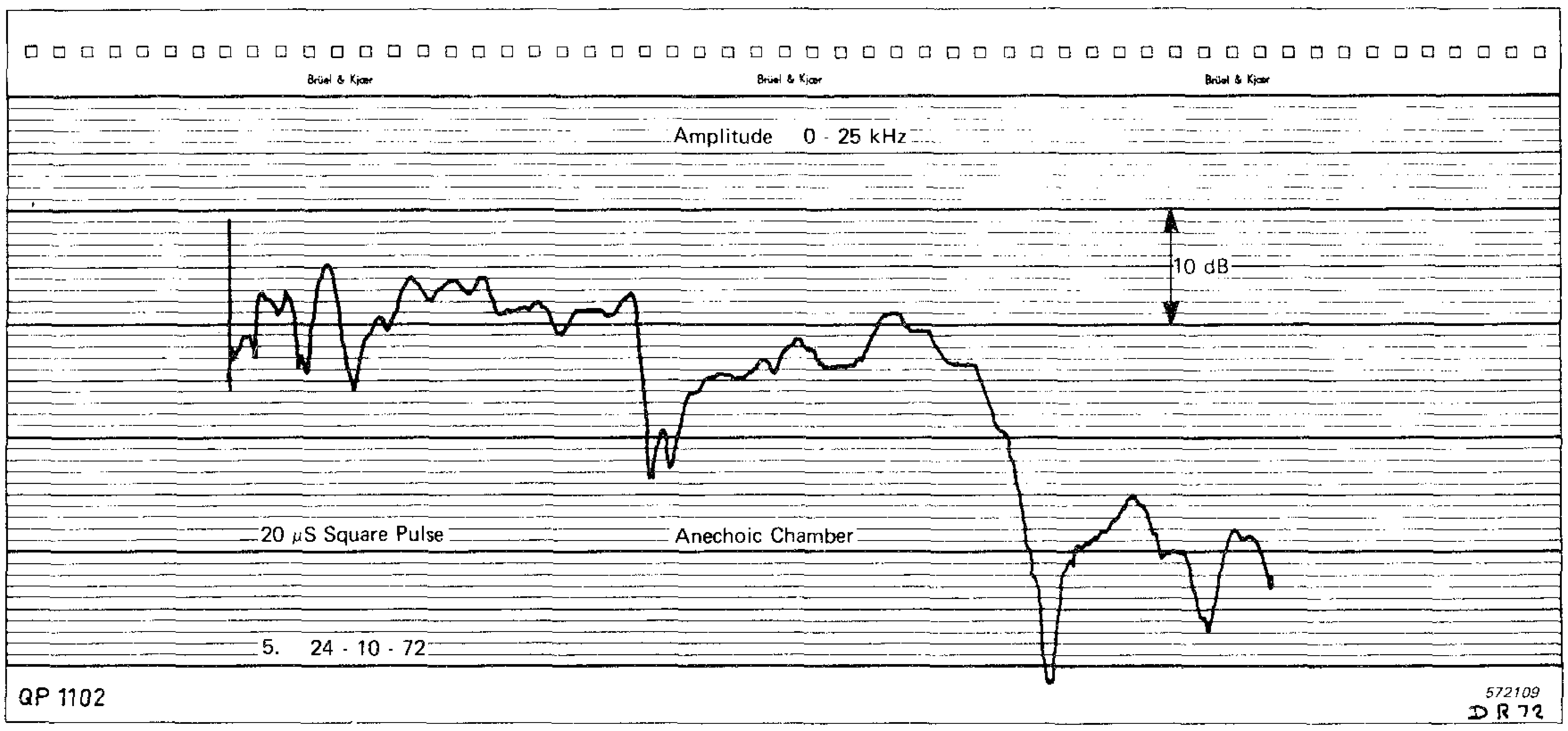

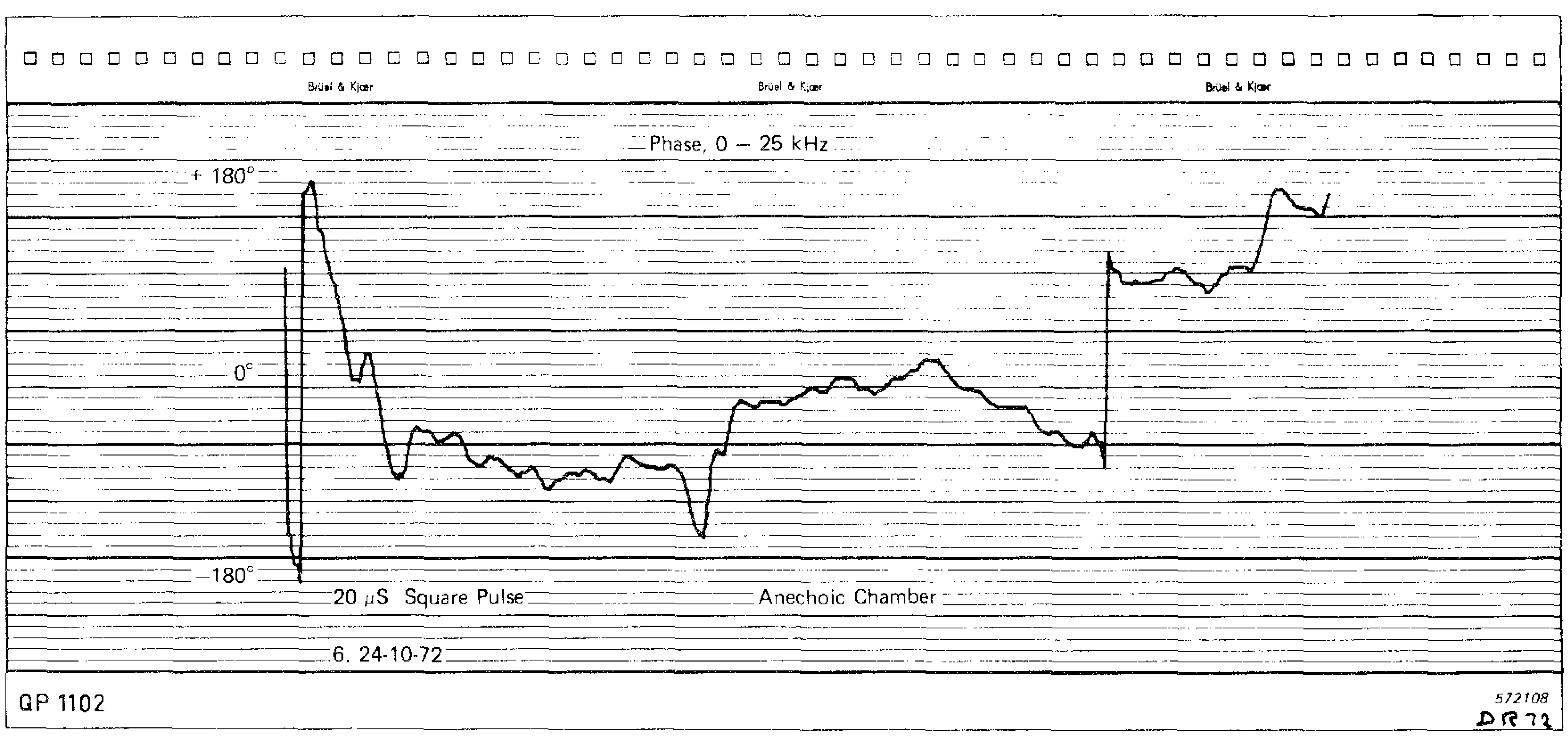

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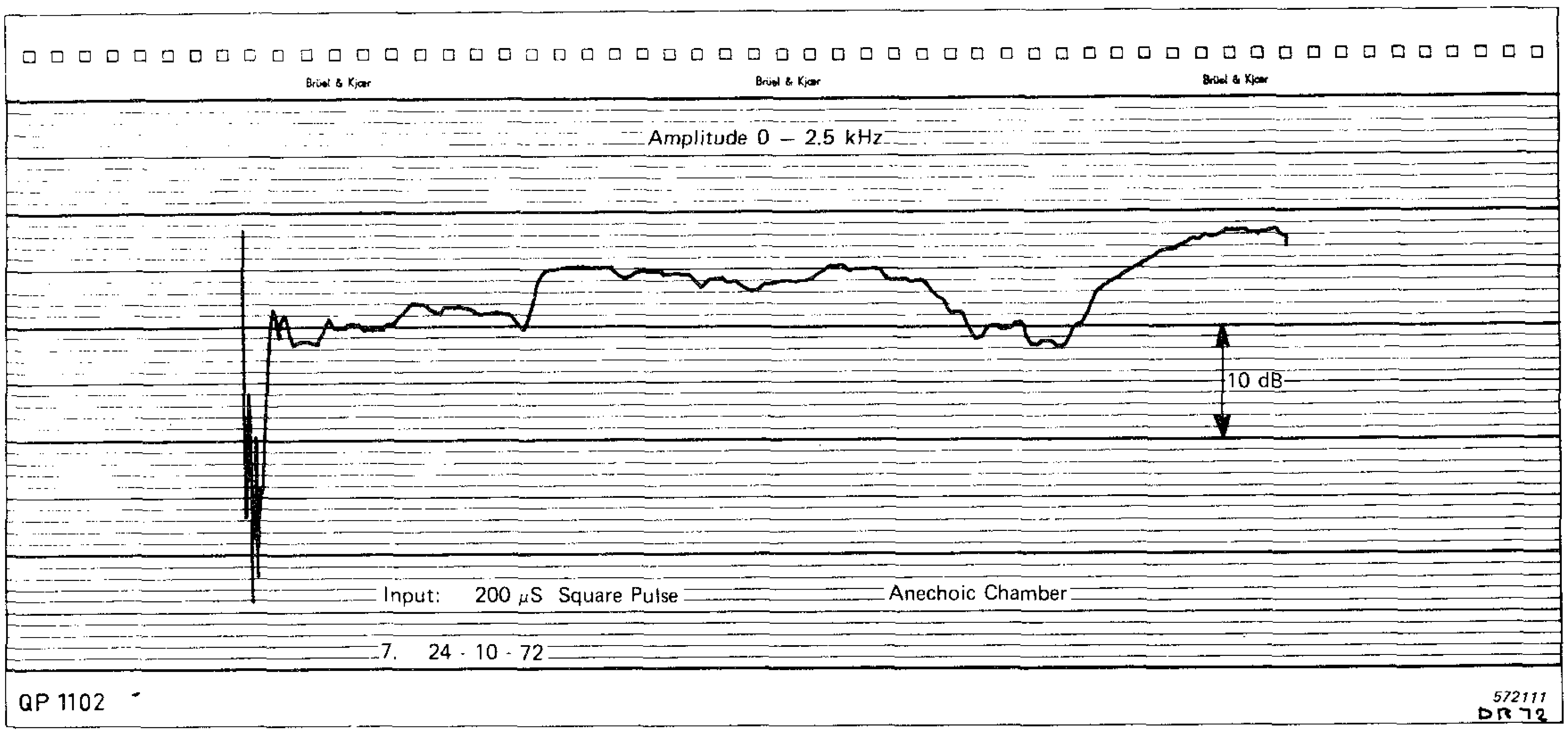

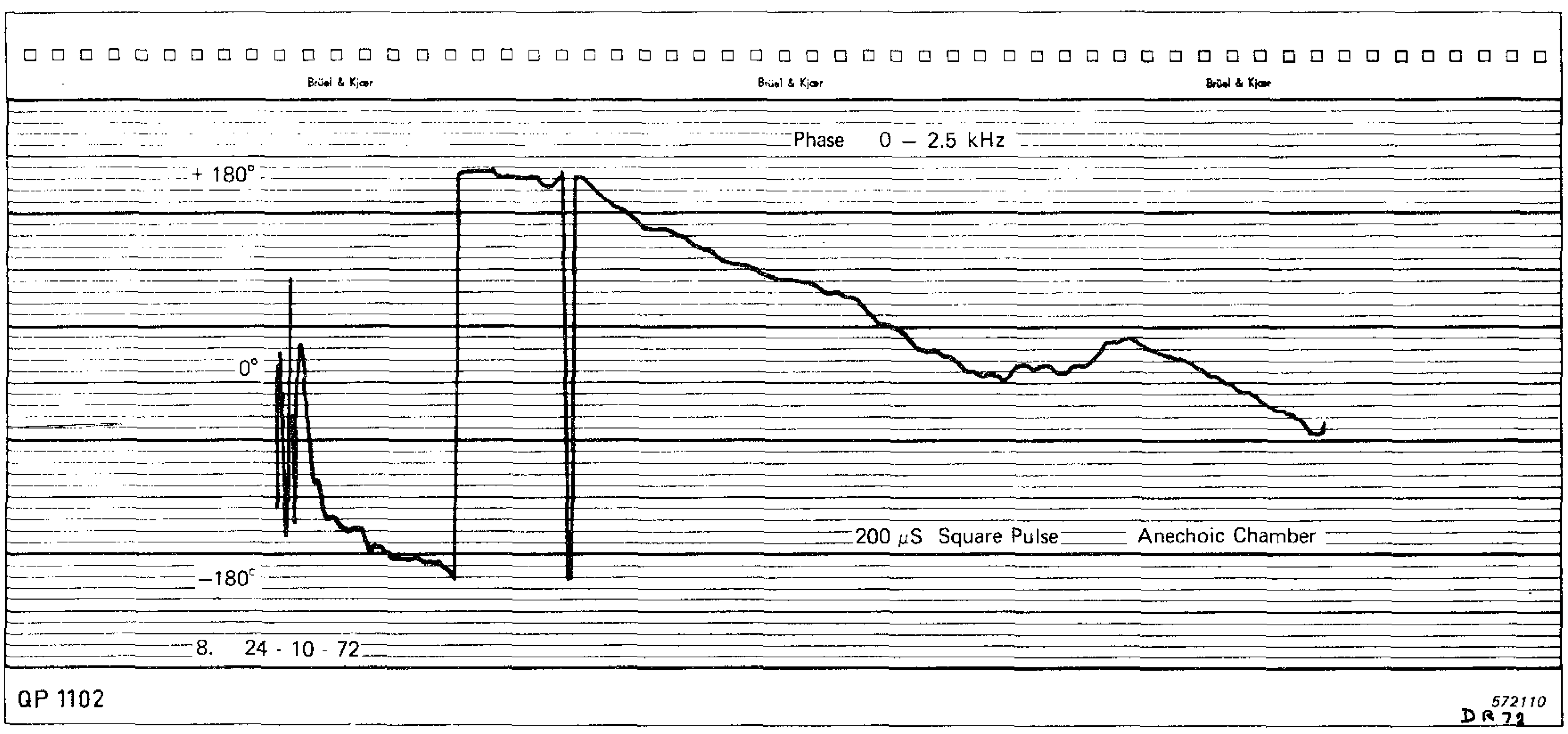

Fig.4. Amplitude and Phase Characteristics measured using 200 µs Square Pulse in Anechoic Chamber

As a comparison, the wide range test was then carried out using the conventional swept sine wave excitation method. The results obtained using this method are given in Fig.5. (Note that they are from 0 – 20 kHz, and not 0 – 25 kHz). Within the bounds of experimental error, agreement with the results of Fig.3 is exact.

All of the above tests were carried out under anechoic conditions. However, one of the advantages of the pulse excitation and FFT method should be that an anechoic chamber is not necessary. It should be possible to carry out the tests in a normal room, provided that it is large enough such that the first reflection from the walls, etc., of the loudspeaker response does not reach the microphone until after the 7502 has captured that response. In order to check this out, further tests were carried out in a normal room.

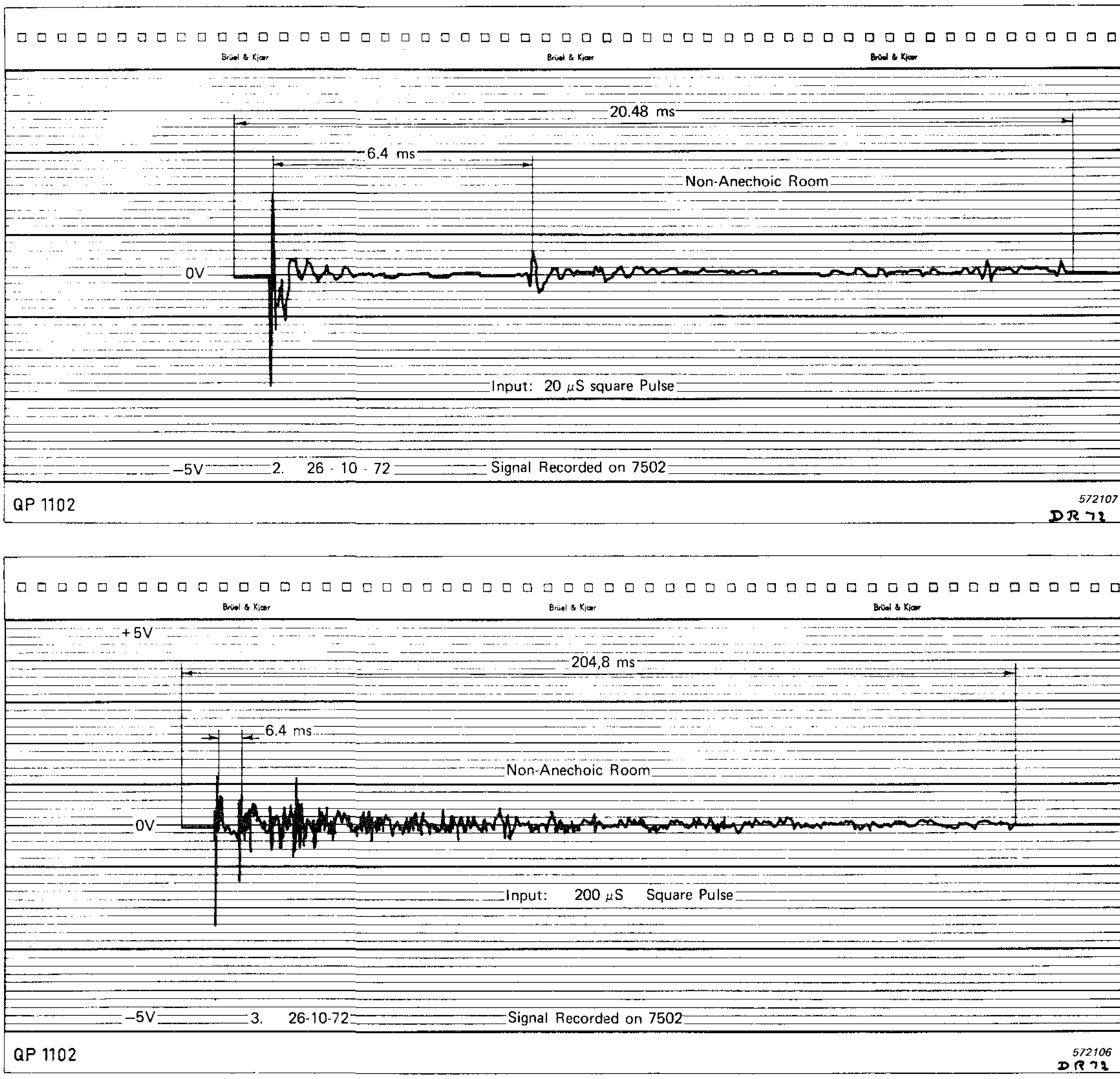

Fig.6. Direct Play Back of Loudspeaker Response to 20 μs and 200 μs pulses in Non-Anechoic Room

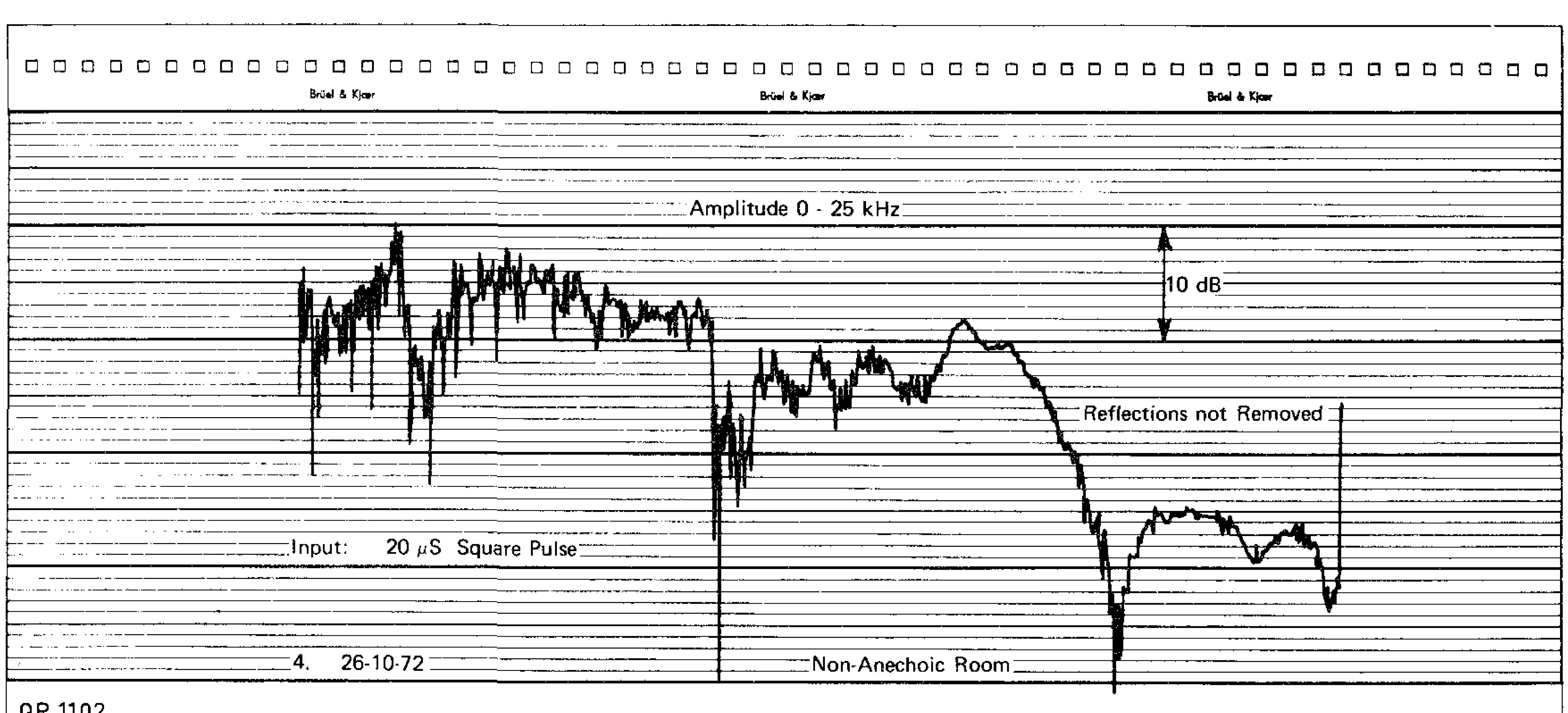

Fig.7. Amplitude Characteristic from complete Loudspeaker Response to 20 μs pulse in Non-Anechoic Room

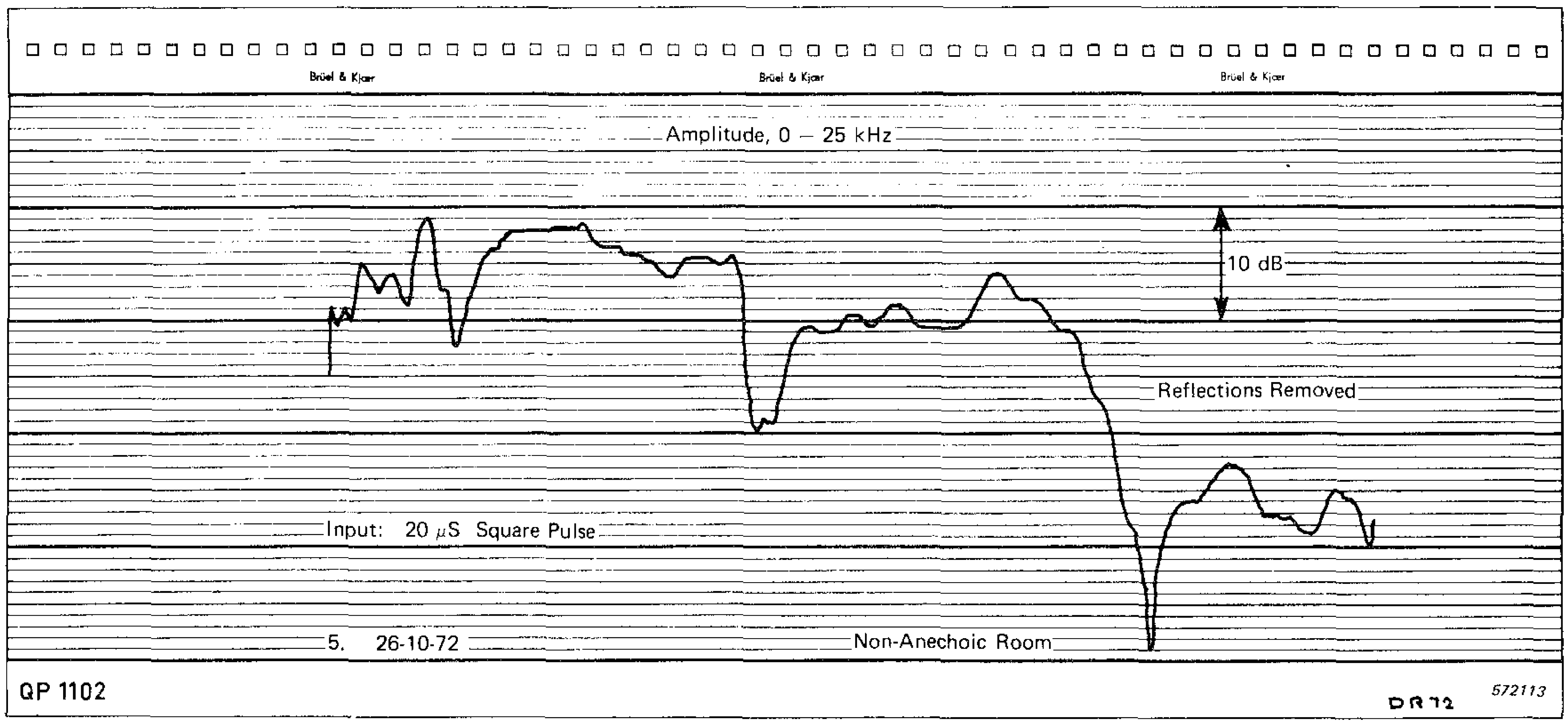

Fig.8. Amplitude Characteristic from Loudspeaker Response to 20 μs pulse in Non-Anechoic Room with Reflections removed

Fig.7 and 8 show the amplitude of the loudspeaker transfer function as measured in this room using a 20 μs pulse (i.e., from 0 to 25 kHz). Fig.7 shows the result which occured when the FFT algorithm was performed on the entire 20 ms time history, including the reflections. The trace shows a great deal of uncertainty. For Fig.8, the time history was multiplied by a time window (by the software) prior to processing, such that the reflections were removed. A much clearer trace results. However, now, the effective length of the recording has been reduced to 6.4 ms, meaning that the bandwidth has increased to 160 Hz, meaning that detail has been lost. This is still a reasonable bandwidth for a test from 0 – 25 kHz, though, and the agreement between Figs.8 and 3 is excellent.

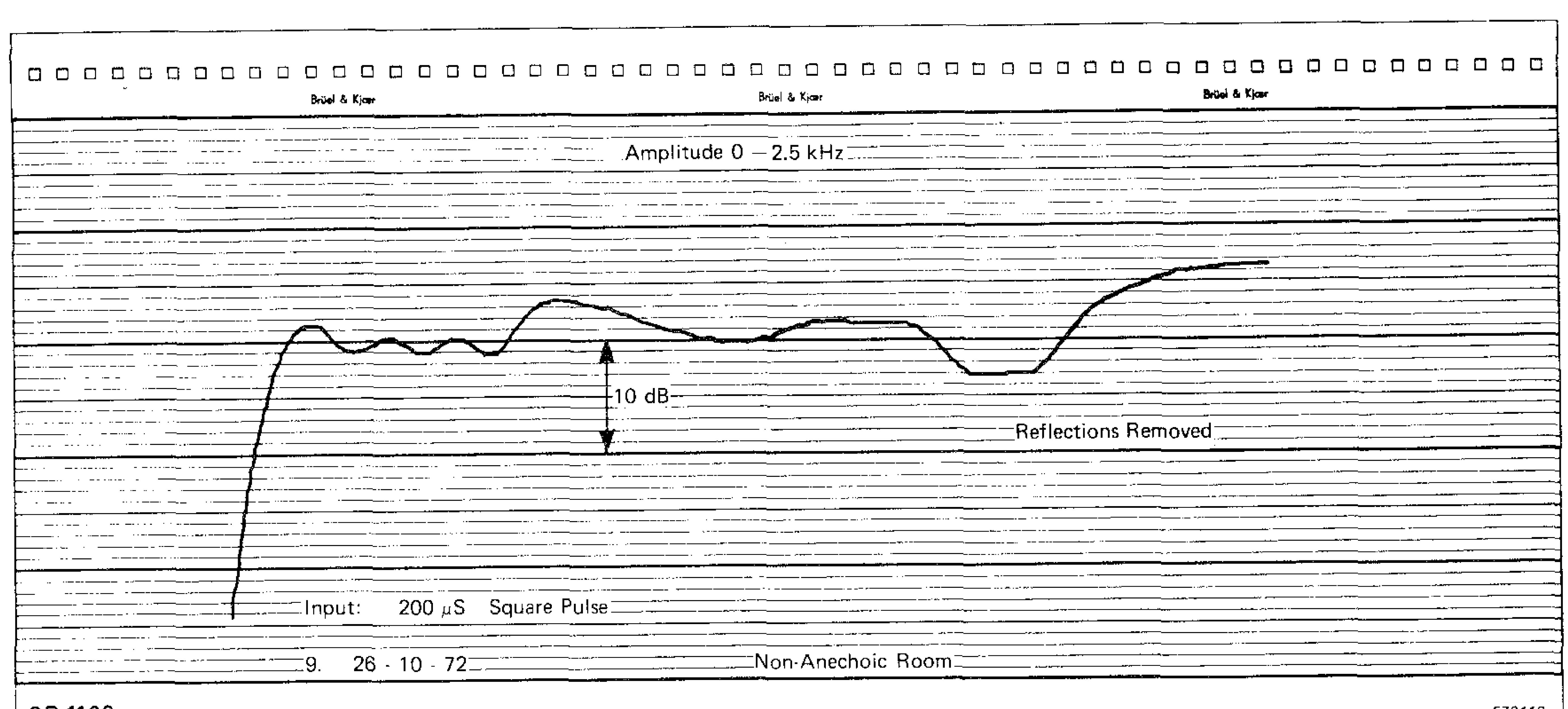

Fig.9. Amplitude Characteristic form Loudspeaker Response to 200 μs pulse in Non-Anechoic Room with Reflections removed

Unfortunately, the same does not apply when a 200 μs pulse is used to measure the transfer function from 0 – 2.5 kHz. The first reflection still appears after 6.4 ms, and hence, when the reflections have been removed, effectively a 6.4 ms time history is again used to obtain the transfer function, giving the same bandwidth of 160 Hz. Now, however, with the range of measurement reduced to 2.5 kHz, this bandwidth is totally inadequate, as the result of Fig.9 shows. Compare Fig.4, which gives the more detailed result. However, to achieve the same degree of agreement between Fig.4 and 9 as is seen between Fig.3 and 8 would require that the first reflection did not appear at the microphone until 64 ms after the start of the response, which in turn would require that the first reflecting surface was at least 20 m away. Hence, its success requires the use of a room approaching concert hall dimensions. This should not be allowed to detract from the success of the method with a 20 μs pulse, though, in a room of much more modest proportions.

All the above tests used ensemble averaging is the time domain to cancel out background noise On each one, 128 runs were averaged out in all, entirely under software control. The effect of the background noise can be seen from Fig.10. This is the trace which was obtained when it was attempted to measure the amplitude of the transfer function of the loudspeaker by the swept sine wave excitation method, in the same room as that which was used to perfrom the tests which produced Figs. 6, 7, 8 and 9.

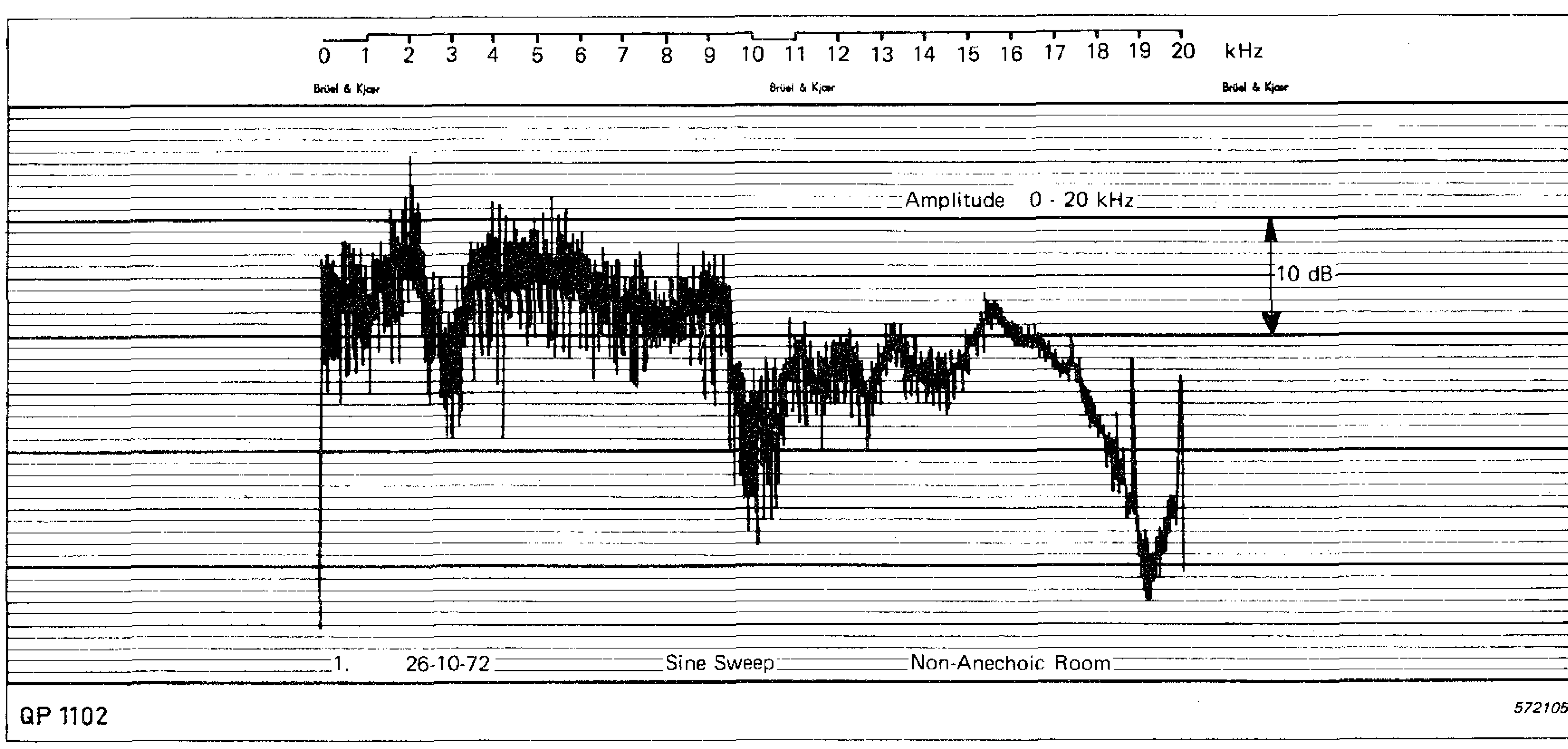

Fig. 10. Amplitude Characteristic measured using 0 → 20 kHz Swept Sine Wave in Non-Anechoic Room

4. CONCLUSIONS

It has been shown that the 7504 and 7502, given appropriate software, can be used as an FFT Analyzer. It has also been shown that an FFT approach can be used to measure Loudspeaker transfer functions which gives equivalent results to those obtained using the conventional swept sine wave excitation method, and which can also be used outside an anechoic chamber where the bandwidth is determined by the room dimension.
Where this method really gains over the conventional one, however, is in flexibility in that, for instance, many runs can be averaged to produce one result, and that the one result contains both amplitude and phase information.

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